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Distribution of surface peaks in 1 + 1 and 2 + 1 ballistic growth models

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Abstract

We present analytical calculations for the average value and the variance of the number of peaks developing at the surface of 1 + 1 and 2 + 1 growing ballistic deposits. Our results are compared with numerical simulations.

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For about 50 years stochastic growth processes have been the focus of much work and interest among physicists. Many different domains such as solid state physics and electrochemistry, but also biology or polymer physics, are involved. An excellent review of this plentiful and diversified activity is given in [1]. During recent decades, problems concerning the shape of a randomly growing surface, especially its width, were more specifically addressed. Detailed numerical studies, scaling relations and, finally, nonlinear stochastic partial differential equations (the most famous one being the celebrated KPZ equation [2]) gave more and more insight into the surface growth problem [3, 4]. In this paper, we will be concerned with the number of peaks, η, appearing at the surface of a growing ballistic deposit. This problem has already been addressed in a previous publication [5], where the average value $\langle\eta\rangle$ was analytically obtained for a 1 + 1 ballistic growth model (BGM). However, we think that a more precise knowledge of the η distribution would be desirable to improve our understanding of the surface morphology. Now, a computation of the variance is still lacking (only an upper bound for its asymptotic value was proposed in [6]). In this paper, we will show how such a computation can be performed and also be extended to a 2 + 1 BGM.

Let us start with the 1+1 problem and specify our model [5]. We begin by taking n columns of unit width each. The centres I_i $(i = 1, \ldots, n)$ of the columns form a one-dimensional lattice we call (I). In the following, large n values and periodic boundary conditions will be understood.

Particles of unit height and of width *slightly larger than one* are successively dropped in randomly chosen columns (with the same probability for all the columns). Assuming that the particles are *impenetrable*, we define $h(i, N)$ as the height of column i after dropping N particles. The surface of the pile is determined by the function $h(i, N)$. An event with $n = 10$ and $N = 18$ is depicted in figure 1(a).

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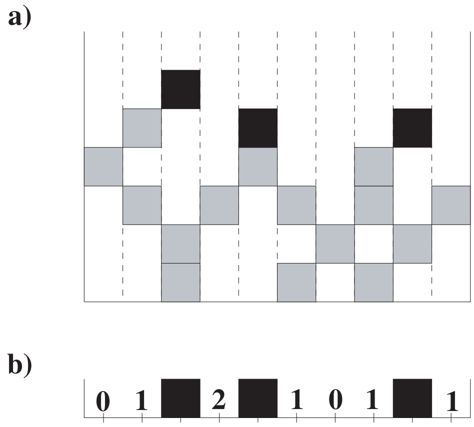

Figure 1. (*a*) The pile in the 1 + 1 BGM (an event is shown with $n = 10$ and $N = 18$). The peaks appear in black. (*b*) The peaks of part (*a*) occupy sites on a one-dimensional lattice. Empty (k-) sites are labelled with their number k of occupied neighbouring sites.

Two particles dropped in neighbouring columns cannot pass 'through' each other and the change of $h(j, N)$ when one extra particle is dropped in column j satisfies the following rule:

$$h(j, N+1) = \max\{h(j-1, N), h(j, N), h(j+1, N)\} + 1 \tag{1}$$

or, equivalently

$$h(j, N+1) = \max_{l \in J} h(l, N) + 1 \tag{2}$$

where J is the set containing j and its nearest neighbours on lattice (I).

A highly rough surface with peaks and valleys develops in the course of particle dropping. j is a peak at some time N if

$$h(j, N) > \max\{h(j-1, N), h(j+1, N)\}. \tag{3}$$

(In figure 1(*a*), peaks are shown in black.) Moreover, looking at the pile from above, we can represent (as shown in figure 1(*b*)) each peak j by an occupied site j on lattice (I). Note that, according to rule (1) two neighbouring sites cannot be occupied. Empty sites can have $k = 0$, 1 or 2 occupied neighbouring sites (see figure 1(*b*)). Let us call them k-sites. If m_k and η are respectively the number of k-sites and of occupied sites (or peaks), we can write the following two relationships:

$$m_0 + m_1 + m_2 = n - \eta \tag{4}$$

$$m_1 + 2\, m_2 = 2\, \eta \tag{5}$$

and, obviously

$$m_0 - m_2 = n - 3\, \eta \tag{6}$$

that will prove to be especially useful.

In the following, we will be mainly interested in the computation of the average value, $\langle \eta \rangle$, and of the variance, var η. Nevertheless, we will see that the latter requires the knowledge of $\langle m_k \rangle$.

For the moment, let us focus on the variation $\Delta\eta$ when we add one particle ($N \to N+1$). Thinking of the rule (1) and also of the definition (3), we see that three possibilities can occur: $\Delta\eta = 0, \pm 1$. For instance, dropping the particle in a column without neighbouring peaks will

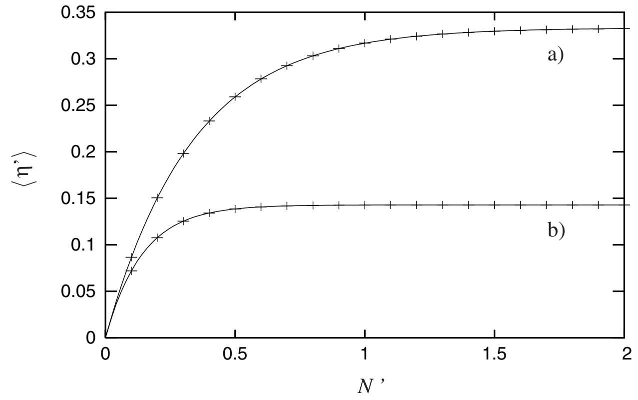

Figure 2. The average value $\langle\eta'\rangle$ of the number of peaks as a function of $N'(\equiv N/n)$: (*a*) 1 + 1 case (points, numerical simulations, 20 000 events with $n = 1000$; curve, equation (12)) (*b*) 2 + 1 case (points, numerical simulations, 50 000 events with $n = 80\,000$; curve, equation (33)).

create a new peak, so $\Delta\eta = +1$. A rapid inspection allows us to write the following recursion relations:

$$\eta \to \eta + 1 \qquad \text{with probability} \quad \frac{m_0}{n} \tag{7}$$

$$\eta \to \eta \qquad \text{with probability} \quad \frac{m_1 + \eta}{n} \tag{8}$$

$$\eta \to \eta - 1 \qquad \text{with probability} \quad \frac{m_2}{n}. \tag{9}$$

Averaging over all possible events, we obtain

$$\Delta\langle\eta\rangle = \left\langle \frac{m_0 - m_2}{n} \right\rangle = 1 - 3\frac{\langle\eta\rangle}{n}. \tag{10}$$

With the rescaled variables $N' \equiv N/n$, $\eta' \equiv \eta/n$, $m'_k \equiv m_k/n$, we obtain, in the limit of large n,

$$\frac{\mathrm{d}\langle\eta'\rangle}{\mathrm{d}N'} = 1 - 3\,\langle\eta'\rangle \tag{11}$$

leading to

$$\langle\eta'\rangle = \tfrac{1}{3}(1 - \mathrm{e}^{-3N'}). \tag{12}$$

This result, already obtained by another method in [5], is compared with simulations in figure 2(*a*). It shows that, asymptotically, about one-third of the columns are peaks. So, the surface of the pile is highly irregular.

For the variance of η, we return to equations (7)–(9) and consider the variation of η^2:

$$\Delta\langle\eta^2\rangle = \frac{1}{n}\langle(\eta+1)^2 m_0 + (\eta)^2(m_1+\eta) + (\eta-1)^2 m_2\rangle - \langle\eta^2\rangle \tag{13}$$

$$= \frac{2}{n}\langle\eta(n - 3\,\eta)\rangle + \frac{1}{n}\langle m_0 + m_2\rangle. \tag{14}$$

With

$$\Delta\langle\eta\rangle^2 = 2\langle\eta\rangle\Delta\langle\eta\rangle + (\Delta\langle\eta\rangle)^2 \tag{15}$$

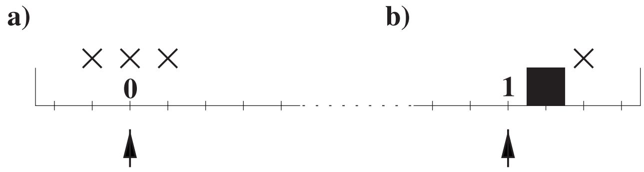

Figure 3. (a) The three ways of destroying a 0-site; if we drop the extra particle (i) in the middle column marked with a cross, then the 0-site is changed into an occupied site, (ii) in the right or left column marked with a cross, then the 0-site is changed into a 1-site. (b) The only one way of creating a 0-site (transformation of a 1- into a 0-site).

we obtain the following differential equation for the variance:

$$\frac{\mathrm{d}\,\mathrm{var}\,\eta'}{\mathrm{d}N'} = -6\,\mathrm{var}\,\eta' + \frac{1}{n}\langle m'_0 + m'_2\rangle - \frac{1}{n}(1 - 3\langle\eta'\rangle)^2. \tag{16}$$

So, we are left with the computation of $\langle m'_0 + m'_2\rangle$.

Let us study the variation of m_k when $N \to N+1$. Figure 3 shows how things go for m_0.

We see that a given 0-site (labelled by an arrow in figure 3(a)) can be destroyed in three ways (if we drop a particle in one of the three columns labelled by a cross). On the other hand, a 1-site can be transformed into a 0-site (see figure 3(b)), only one possibility). So, we deduce

$$\Delta\langle m_0\rangle = \frac{1}{n}\langle\, -3\,m_0 + m_1\,\rangle. \tag{17}$$

In the same way, we obtain

$$\Delta\langle m_1\rangle = \frac{1}{n}\langle\, 2\,\eta + 2\,m_0 - 3\,m_1 + 2\,m_2\,\rangle = \frac{1}{n}\langle\, 2\,m_0 - 2\,m_1 + 4\,m_2\,\rangle \tag{18}$$

$$\Delta\langle m_2\rangle = \frac{1}{n}\langle\, m_1 - 3\,m_2\,\rangle. \tag{19}$$

It is easy to check that

$$\Delta\langle m_0\rangle + \Delta\langle m_1\rangle + \Delta\langle m_2\rangle = -\,\Delta\langle\eta\rangle \tag{20}$$

$$\Delta\langle m_1\rangle + 2\,\Delta\langle m_2\rangle = 2\,\Delta\langle\eta\rangle \tag{21}$$

in agreement with equations (4), (5).

Introducing the 3-vector m' of components m'_0, m'_1 and m'_2, equations (17)–(19) lead to the following set of differential equations:

$$\frac{\mathrm{d}\langle m'\rangle}{\mathrm{d}N'} = A\langle m'\rangle \tag{22}$$

where A is a (3×3) matrix:

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 2 & -2 & 4 \\ 0 & 1 & -3 \end{pmatrix}.$$

Solving (22) with the initial conditions $m'_0 = 1, m'_1 = m'_2 = 0$ at $N' = 0$, we obtain

$$\langle m'_0\rangle = \tfrac{1}{15}(2 + 10\,\mathrm{e}^{-3N'} + 3\,\mathrm{e}^{-5N'}) \tag{23}$$

$$\langle m'_1\rangle = \tfrac{2}{5}(1 - \mathrm{e}^{-5N'}) \tag{24}$$

$$\langle m'_2\rangle = \tfrac{1}{15}(2 - 5\,\mathrm{e}^{-3N'} + 3\,\mathrm{e}^{-5N'}). \tag{25}$$

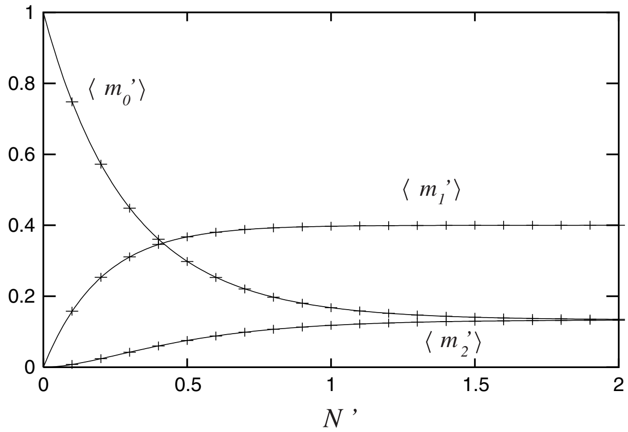

Figure 4. For the 1 + 1 BGM, the average value of the number of k-sites m'_k, $k = 0, 1, 2$, as a function of N' (points, numerical simulations, 20 000 events with $n = 1000$; full curves, equations (23)–(25)). For further explanations, see the text.

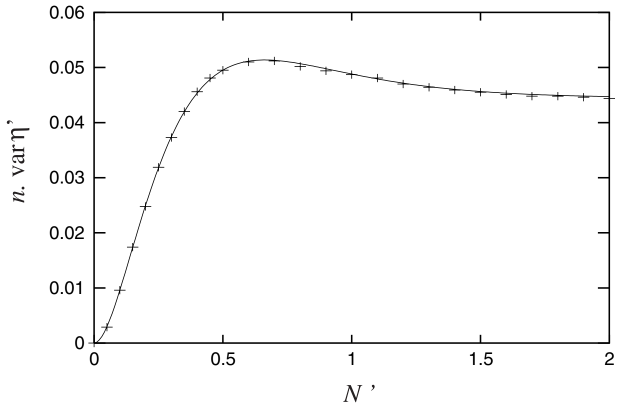

Figure 5. For the 1 + 1 BGM, the variance of η' (multiplied by n) as a function of N' (points, numerical simulations, 20 000 events with $n = 1000$; full curve, equation (26)).

Returning to equation (16), we obtain for the variance

$$\operatorname{var}\eta' = \frac{1}{45n}(2 + 5\,\mathrm{e}^{-3N'} + 18\,\mathrm{e}^{-5N'} - (45\,N' + 25)\mathrm{e}^{-6N'}) \tag{26}$$

$$\to \frac{2}{45n} \qquad \text{when} \quad N' \to \infty. \tag{27}$$

We conclude that, for large n values, the distribution of η' is practically a δ function.

All these results are shown to agree quite well with our computer simulations (see figures 4 and 5). Note that the variance exhibits a maximum for $N' \approx 0.7$ i.e. when the system is not completely 'filled'.

Now, we show how the previous basic idea can be applied to a 2 + 1 BGM. Let us consider hexagonal columns (the horizontal section of a given column is a regular hexagon H). The new lattice (I) (centres of the column bases) is, this time, triangular. Impenetrable particles of unit height and size slightly larger than H are dropped in randomly chosen columns. So, we keep

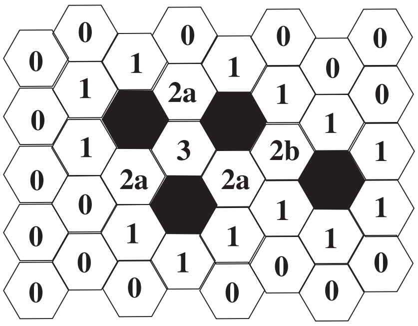

Figure 6. The two-dimensional analogue of figure 1(b). The occupied sites (peaks) are black hexagonal cells. Empty cells are labelled with their number of neighbouring occupied cells. Note the peculiarities of the 2-cells. (For the sake of clarity, the underlying triangular lattice (I) is not shown.)

the same actualization rule, equation (2). As before, j is a peak if

$$h(j, N) > \max_{l \in J^*} h(l, N) \tag{28}$$

where J^* is the set of nearest neighbours of j on lattice (I).

The peaks are now represented by occupied cells, the centres of which belong to the triangular lattice (I). An event is depicted in figure 6, where the pile is seen from above (the peaks are shown in black). Empty cells can have $k = 0, 1, 2$ or 3 occupied neighbouring cells. It is worthwhile to notice that the 2-sites can appear in two topologically different situations ($2a$ and $2b$). With the same notations as before (n, total number of columns; m_k, number of k-sites, $m_2 \equiv m_{2a} + m_{2b}$), the relations (4), (5) become

$$m_0 + m_1 + m_2 + m_3 = n - \eta \tag{29}$$

$$m_1 + 2\,m_2 + 3\,m_3 = 6\,\eta. \tag{30}$$

The variation of η when $N \to N + 1$ is still given by (7)–(9) except for adding the new possibility

$$\eta \to \eta - 2 \qquad \text{with probability} \quad \frac{m_3}{n}. \tag{31}$$

Now, it is easy to write

$$\Delta\langle\eta\rangle = 1 - 7\frac{\langle\eta\rangle}{n} \tag{32}$$

$$\langle\eta'\rangle = \tfrac{1}{7}(1 - \mathrm{e}^{-7N'}). \tag{33}$$

(A comparison of this result with a numerical simulation is made in figure 2(b).)

Straightforward algebra leads to the following equation for the variance:

$$\frac{\mathrm{d}\,\mathrm{var}\,\eta'}{\mathrm{d}N'} = -14\,\mathrm{var}\,\eta' + \frac{1}{n}\langle m_0' + m_2' + 4\,m_3'\rangle - \frac{1}{n}(1 - 7\langle\eta'\rangle)^2. \tag{34}$$

To compute the average values $\langle m_k'\rangle$, we must first write the recursion relations. This is done by splitting m_2' into m_{2a}' and m_{2b}' and introducing the 5-vector m' of components m_0', m_1', m_{2a}', m_{2b}' and m_3'. Close inspection leads to

$$\frac{\mathrm{d}\langle m'\rangle}{\mathrm{d}N'} = B\langle m'\rangle \tag{35}$$

with the (5×5) B matrix

$$B = \begin{pmatrix} -7 & 3 & 0 & 0 & 0 \\ 6 & -6 & 9 & 8 & 3 \\ 0 & 2 & -11 & 4 & 9 \\ 0 & 1 & 2 & -11 & 3 \\ 0 & 0 & 1 & 0 & -13 \end{pmatrix}.$$

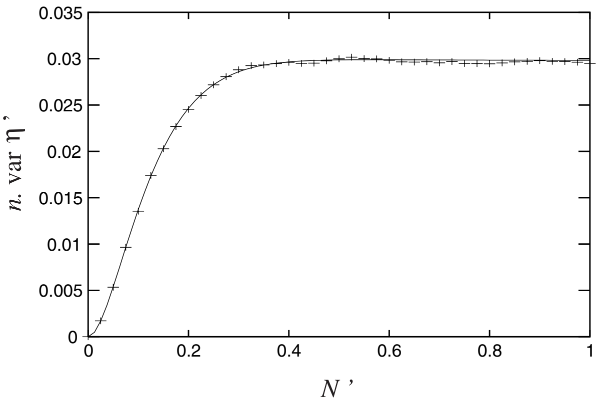

Figure 7. The same as figure 5 but for the 2 + 1 BGM (full curve, equation (36); points, numerical simulations, 50 000 events with $n = 80\,000$).

(Note that we will not obtain such linear relations if we do not split m_2'.)

It is easy to check that (35) is consistent with (29) and (30).

Solving equations (35) and (34), we obtain the variance expression:

$$\operatorname{var}\eta' = \frac{1}{n}\left(\frac{19}{637} + \frac{e^{-7N'}}{245} + \frac{e^{-12N'}}{5} + \frac{2\,e^{-13N'}}{13} - \left(N' + \frac{247}{637}\right)e^{-14N'}\right). \quad (36)$$

This last result is displayed in figure 7.

We observe that, for both models (1 + 1 and 2 + 1), the quantities $\langle\eta'\rangle$ and $n \cdot \operatorname{var}\eta'$ are only functions of N' ($\equiv N/n$). Let us check that this is still true if we change the shape of the columns in the 2 + 1 model (provided that we keep the same rules (2) and (28)).

For instance, for a lattice (I) of coordination z, we will obtain analytically

$$\langle\eta'\rangle = \frac{1}{z+1}(1 - e^{-(z+1)N'}) \quad (37)$$

$$\frac{d\operatorname{var}\eta'}{dN'} = -2\,(z+1)\operatorname{var}\eta' + \frac{1}{n}\left\langle\sum_{k=0}^{\infty}(k-1)^2 m_k'\right\rangle - \frac{1}{n}(1-(z+1)\langle\eta'\rangle)^2. \quad (38)$$

(Of course, the series on the right-hand side only contains a finite number of terms.)

$\langle m_k'\rangle$ is obtained by solving a linear differential equation of type (35). Obviously, the B matrix depends on the lattice (I). In particular, its dimension is determined by topological considerations as previously discussed.

Nevertheless, this is enough to convince oneself that the quantity $n \cdot \operatorname{var}\eta'$, solution of (38), is only a function of N'. Note that the asymptotic value ($N' \to \infty$) is easily obtained from (38):

$$n\operatorname{var}\eta' = \frac{1}{2\,(z+1)}\left(\sum_{k=0}^{\infty}(k-1)^2\langle m_k'\rangle\right) \quad (39)$$

$\langle m_k'\rangle$ being taken at $N' = \infty$.

In summary, we have computed the surface peak distribution for the 1 + 1 and 2 + 1 BGMs. Note that the local minima on the surface have the same distribution as the peaks in the 1 + 1 BGM (the numbers of local minima and peaks are the same in that case). However, for the 2 + 1 BGM, the study of local minima requires more elaborate considerations. It would also be fine to study along the same lines more fundamental properties of growing surfaces such as, for instance, the roughness or the mean height of the pile. Unfortunately, the surface width is related to the depth of the valleys and we rapidly realize that an exact treatment leads to highly nonlinear coupled equations (the same problem for the mean height). Can those equations be simplified in some limiting cases? This is still an open question that is far beyond the scope of this paper.

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